A novel approach to evaluate the elastic impact of spheres on thin plates

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HIGHLIGHTS

- Description of Zener’s approach to evaluate impact energy characteristics.
- Derivation of a Zener model based simple analytical approach for the CoR.
- Evaluation of the influence of the ratio of sphere diameter to plate thickness.
- Determination of the influence of impact velocity on the CoR.

GRAPHICAL ABSTRACT

ABSTRACT

A novel analytical approach to solve the otherwise mathematically strenuous Zener model for elastic sphere impacts on large, thin plates, has been presented to find a comfortable solution for the coefficient of restitution (CoR). The proposed analytical approach provides accurate results for the range of coefficients of restitution larger than 0.2 and gives a very good approximation of the Zener model. Furthermore, the Zener model has been numerical solved with high accuracy and has been used to evaluate the inelasticity parameter as well as the coefficient of restitution for different material combinations and ratios of sphere diameter to plate thickness. Both approaches have been used to evaluate the coefficient of restitution of elastic glass beads at impact with glass plates of different thicknesses using experimental free fall test measurements. A significant dependence of the coefficient of restitution of elastic spheres on the ratio of sphere diameter to plate thickness as well as on the impact velocity and on the inelasticity parameter has been observed respectively, which can be well described by the Zener model and the proposed analytical approach.

1. Introduction

An impact is an energetically interactive dynamic collision that occurs between two bodies within a short time period. It consists of an initial instant known as incidence \( t = t_0 \), where the colliding bodies (with some relative velocity) come in contact, followed by a short time period known as the compression phase \( t_0 \leq t \leq t_A \), where the gradually increasing contact force (de-acceleration) reaches a maximum such that the velocities of the colliding bodies are reduced to zero, followed by a short time period known as the restitution or rebound (de-compression) phase \( t_A \leq t \leq t_R \), where the stored elastic energy is released and converted into kinetic energy causing a rebound of the colliding bodies. Impacts are characterized by a change in the state of motion, the impulse and the energy of the colliding bodies while the momentum is conserved at all times. Since only internal forces are transmitted at impact, the total impulse and the total energy are conserved. The laws of impact have been presented by Christiaan Huygens (1629–1695) (Fassmann et al., 1974).
Natural impacts occur for instance, in astronomy during aggregation and agglomeration of gas, dust and particles in protoplanetary disks as well as in interstellar clouds. Furthermore, natural impacts occur in physical geography, during ice and debris avalanches and during weather phenomenon like tornados and dust devils. In industrial processes, impacts occur frequently during processing, transportation and handling of particulate raw materials as well as products. Considering only process engineering endeavors, impacts of particle products like agglomerates and granules (between themselves and with walls of the apparatuses) occur frequently during unit operations such as crushing, milling, fluidization, apparatus filling, hopper discharge, pneumatic conveying, coating, mixing, etc. Thus, studying the impact behavior of particle products are vitally necessary to design processing apparatuses and corresponding unit operations to assure excellent product quality.

2. Energy absorption during impact

In general, the energy aspects of an impact event are quantified using the coefficient of restitution (CoR) \( e \), which is a physical parameter representing the ratio of the impulses of the rebound phase \( (t_{A} \leq t \leq t_{B}) \) and the compression phase \( (0 \leq t \leq t_{C}) \)

\[
e = \frac{\int F(t) \, dt}{\int F_{eq}(t) \, dt}
\]

with the acting contact force \( F(t) \). Provided that a merely translatory motion appears, the CoR can be generally specified as the ratio of the relative velocities of the colliding bodies 1 and 2 before and after impact \( (\Delta v_{1A}, \Delta v_{2A}) \) as long as there is no change in the direction of sliding during impact

\[
e = \frac{\left| \frac{1}{\sqrt{E_{1}}} \frac{\Delta v_{2g}}{\sqrt{\frac{1}{2} m_{1} \cdot \frac{\Delta v_{1g}}{\sqrt{\frac{1}{2} m_{2} \cdot \Delta v_{2g}}}} \right|}{\left| \frac{1}{\sqrt{E_{2}}} \frac{\Delta v_{1g}}{\sqrt{\frac{1}{2} m_{1} \cdot \frac{\Delta v_{1g}}{\sqrt{\frac{1}{2} m_{2} \cdot \Delta v_{2g}}}}} \right|}
\]

For a perfectly elastic impact, the CoR has a characteristic value of \( e = 1 \), while for elastic–plastic and perfectly plastic impacts assumes characteristic values of \( 0 < e < 1 \) and \( e = 0 \), respectively. During impact of a body (particle) against a rigid, stationary surface or plate, Eq. (2) reduces to

\[
e = \left| \frac{\frac{1}{\sqrt{E_{1}}} \frac{\Delta v_{2g}}{\sqrt{\frac{1}{2} m_{1} \cdot \frac{\Delta v_{1g}}{\sqrt{\frac{1}{2} m_{2} \cdot \Delta v_{2g}}}}} \right| \Bigg/ \left| \frac{1}{\sqrt{E_{2}}} \frac{\Delta v_{1g}}{\sqrt{\frac{1}{2} m_{1} \cdot \frac{\Delta v_{1g}}{\sqrt{\frac{1}{2} m_{2} \cdot \Delta v_{2g}}}}} \right|
\]

Thus, the CoR is predominantly determined using free fall tests (see Antonyuk et al. (2010), Goldsmith (1960), Iveson and Litster (1998), and Sondergaard et al. (1990)). In doing so, spherical particles are dropped freely from a desired height onto a rigid surface and the normal or oblique impact is thereby analyzed. The procedures of impact and rebound are usually recorded using a high-speed camera (for example, see Antonyuk et al. (2010) and Kharaz et al. (1999)). From the recorded pictures, the impact and the rebound velocities can be determined and the CoR may be thereby calculated with Eq. (3). Other methods include the determination of the rebound height using a camera (as done in Tillett (1954)), where the CoR may be calculated according to

\[
e = \left| \frac{\frac{1}{\sqrt{E_{1}}} \frac{\Delta v_{2g}}{\sqrt{\frac{1}{2} m_{1} \cdot \frac{\Delta v_{1g}}{\sqrt{\frac{1}{2} m_{2} \cdot \Delta v_{2g}}}}} \right| \Bigg/ \left| \frac{1}{\sqrt{E_{2}}} \frac{\Delta v_{1g}}{\sqrt{\frac{1}{2} m_{1} \cdot \frac{\Delta v_{1g}}{\sqrt{\frac{1}{2} m_{2} \cdot \Delta v_{2g}}}}} \right|
\]

(\( h_{a} \) initial height of fall or initial drop height, \( h_{R} \) rebound height), or the measurement of the time interval between two consecutive impacts of one particle by evaluation of the acoustics of the impact (as reported in (Bernstein, 1977; Higa et al., 1996; Hofstee, 1992; Yin-Chao et al., 1970)). In this case, neglecting the air resistance, the impact and rebound velocities are determined from the time interval (time of flight) \( \Delta t \) between the instants of an impact \( t = t_{A1} \) and its subsequent impact \( t = t_{A2} \)

\[
v = \frac{g \Delta t}{2}
\]

With Eq. (5), the CoR (Eq. (3)) thus reduces to

\[
e = \frac{\Delta t_{A1}}{\Delta t_{n}}
\]

Apart from the mechanistic approaches outlined above, there exist several constitutive approaches for the theoretical approximation of the CoR, that are based on different physical assumptions regarding the material behavior. The perfectly elastic impact, having a CoR of \( e = 1 \), has been described by Hertz (Hertz, 1881), at which the approach for the normal elastic force \( F_{el} \) is as follows

\[
F_{el} = \frac{2}{3} E_{1,2} \pi R_{1,2}^{5/3}
\]

where \( s \) is the normal elastic displacement of the contact sphere and the plate and \( E_{1,2} \) is the effective modulus of elasticity according to Tomas (Tomas, 2007)

\[
E_{1,2} = 2 \left( \frac{1 - v_{1}^{2}}{E_{1}} + \frac{1 - v_{2}^{2}}{E_{2}} \right)^{-1} \approx \frac{2}{1 - v_{2}^{2}} E_{1}, \text{ for } E_{2} \gg E_{1}
\]

with the moduli of elasticity \( E_{1} \) and \( E_{2} \) and the Poisson's ratios \( v_{1} \) and \( v_{2} \) of the impacting bodies. The effective radius \( R_{1,2} \) of surface curvature results from the radius of surface curvature of the contact areas of the impacting bodies before any contact flattenings \( R_{1}, R_{2} \)

\[
R_{1,2} = \left( \frac{1}{R_{1}} + \frac{1}{R_{2}} \right)^{-1} \approx R_{1}, \text{ for } R_{2} \to \infty
\]

Analytical results evaluated using the Hertz model (Hertz, 1881) for elastic impacts suffer from inaccuracies, as it completely ignores dissipation of energy during impact by viscous contributions (typically seen in case of viscoelastic solids) and the always existing elastic wave propagation. However, on exceeding the yield velocity \( v_{y} \), when viscous material behavior becomes significant (typically seen in case of viscoelastic–viscoplastic or elastic–viscoplastic solids) or when the influences of elastic wave propagation become significant, the Hertz model (Hertz, 1881) becomes almost completely invalid. In literature (see (Walton, 1993; Stronge, 2000; Brilliantov et al., 1996; Pöschel and Luding, 2001; Thornton, 1997)), one can find several theoretical models describing impact behavior at these conditions.

In case an impact between two spherical elastic bodies takes place, during and after impact, these bodies often do not behave like perfect rigid bodies as presupposed by the Hertz theory (Hertz, 1881). During impact, a pressure develops at the contact area, and a stress wave arises due to the local deformation, and propagates inwards through the bodies away from the point of excitation (located at the centre of the contact area). Moreover, the generation of surface and body seismic waves produces an energy loss in the region of impact. The elastic waves propagate through the solid bodies exhibiting a characteristic velocity, which are refracted or reflected at interfaces and may propagate back to the point of excitation, where they cause additional energy dissipation. Thus, multiple wave reflections may increasingly affect the energy dissipation associated with the impact event.

According to experimental results and theoretical models of several authors (Yin-Chao et al., 1970; Stronge, 2000; Raman, 1920; Zener, 1941; Hunter, 1957; Reed, 1985; Koller, 1983), a considerable content of the kinetic energy of impact can be transformed into elastic waves propagating through the solid bodies (see also Tillett, 1954). The approach of Zener (Zener, 1941) is based on the assumption that at excitation of large, thin plates during normal elastic impact, the kinetic energy of impact is
divided between setting up a Hertzian stress field at the contact area and the radial propagation of (flexural) elastic waves. Hence, the model of Zener is valid for the impact between a sphere and a large, thin plate, at which the contact time is less than twice the transition time of a flexural wave across the dimension of the plate.

For the calculation of the rebound velocity \( v_b \) of the sphere and consequently the CoR \( e \), the equations of motion of the sphere and the plate are derived and solved after setting the contact. The motion of the sphere is described by Newton 2nd law of motion. The equation of motion of the plate gives the proportionality between the displacement at the contact and the transmitted impulse (Koller, 1983). Both differential equations are combined and for the elastic impact, it follows

\[
\frac{d^2 s}{dt^2} + \frac{1}{m_{1,2}} F(s) + \alpha \frac{dF(s)}{dt} = 0, \tag{10}
\]

with \( s \) the displacement resulting from the difference of the displacement of the center of the sphere and the center of the plate, \( t \) contact time, \( m_{1,2} \) the effective (averaged) mass

\[
m_{1,2} = \frac{m_1 m_2}{m_1 + m_2} = \left( \frac{1}{m_1} + \frac{1}{m_2} \right)^{-1}, \tag{11}
\]

\( F \) the contact force on basis of the Hertz law (Hertz, 1881), \( \alpha \) a constant

\[
\alpha = \sqrt{\frac{k_0}{12E_2}} = \frac{1}{16 \cdot \rho_2 R_2^3} \tag{12}
\]

\( H = 2h_p \) the plate thickness, \( \rho_2 \) the density of the plate and \( E_2 \) the modulus of elasticity of the plate. The initial conditions are as follows:

\[
s = 0 \quad \text{at} \quad t = 0. \tag{13}
\]

The equation (10) is then transformed into a dimensionless differential equation

\[
\frac{d^2 \sigma}{d \tau^2} + \left( 1 + \lambda \frac{d}{d \tau} \right) \sigma^{3/2} = 0 \tag{14}
\]

with the dimensionless relative displacement

\[
\sigma(\tau) = \frac{s(\tau)}{F \cdot V_{th}} \tag{15}
\]

and the dimensionless time

\[
\tau = \frac{t}{T} \tag{16}
\]

where \( t \) is the time and \( T \) is a characteristic material constant with the dimension of time. During the transformation of equation (10) into (14), this constant \( T \) results to

\[
T = \left( \frac{m_{1,2}}{k \cdot V_{th}^2} \right)^{2/5} \tag{17}
\]

which is

\[
T = 0.311 \cdot T_{el} \tag{18}
\]

approximately one-third of the elastic contact time of impact calculated by Hertz for the case of a plate of infinite thickness. \( V_{th} \) gives the initial impact velocity and \( k \) is the Hertzian stiffness constant

\[
k = \frac{2}{3} E \cdot R_1^{1/2}. \tag{19}
\]

The parameters that influence the impact, mainly characterizes the compliance of impact partners, are summarized in the dimensionless inelasticity parameter

\[
\lambda = \frac{\pi^{3/2}}{3^{1/2}} \left( \frac{R}{2h_p} \right)^2 \left( \frac{V_{th}}{c} \right)^{1/5} \left( \frac{\rho_1}{\rho_2} \right)^{3/5} \left( \frac{E_1}{(1 - \nu_1^2)} + \frac{E_2}{(1 - \nu_2^2)} \right)^{2/5}, \tag{20}
\]

with \( \lambda \) the propagation velocity of quasi-longitudinal waves in thin plates (see Cremer and Heckl (1996))

\[
\lambda' = \sqrt{\frac{E_s}{(1 - \nu_s^2)}} \rho_2 \tag{21}
\]

\( \rho_1, \rho_2 \) the densities of the sphere (particle) and the plate, respectively, and \( \gamma \) the ratio of sphere (particle) diameter \( d = 2R \) to plate thickness \( H = 2h_p \) (Sondergaard et al., 1990)

\[
\gamma = \frac{d}{H} = \frac{R}{h_p} \tag{22}
\]

Using the initial conditions

\[
\sigma = 0 \quad \text{at} \quad \tau = 0, \quad \frac{d \sigma}{d \tau} = 1 \quad \text{at} \quad \tau = 0. \tag{23}
\]

the differential Eq. (14) can be numerically solved. Independent of the model of impact, the exact contact time can be predicted when the (Newtonian) force between the two bodies becomes again zero. Since at the model of Zener all forces are equal (Eq. (7)), all the forces will become zero when the displacement \( s \) goes back to zero. Now, the CoR reflects the absolute value of the dimensionless rebound velocity \( d\sigma/d\tau \) at the dimensionless contact time \( \tau_{el} \)

\[
e = \left| \frac{d\sigma}{d\tau} \right| \sigma = 0 \ne 0 \quad \frac{V_T \cdot \sigma}{V_{th}} = \left( \frac{d\sigma}{d\tau} \right|_{\sigma = 0 \ne 0} \tag{24}
\]

3. Proposed analytical approach

Below, the nonlinear differential equation put forward by Zener (Zener, 1941) Eq. (10) is transformed to be accessible for an analytical solution by simplifying the force–displacement \( F(s) \) relation of the Hertz Eq. (7), so that the time-dependent displacement \( s(t) \) is merely linear in the equation ( \( k \) Hertzian stiffness constant acc. to Eq. (19))

\[
F(s) = k \sqrt{S_b} s(t), \tag{25}
\]

where the characteristic displacement during impact

\[
s_c = TV_A \cdot \sigma_c \tag{26}
\]

with \( \sigma_c = 1 \), is independent of the time, while \( s(t) \) follows the model of Zener according to Eq. (15). Likewise, it is possible to express Eq. (25) in dependence on the maximum displacement at elastic impact according to Hertz (Hertz, 1881)

\[
s_{max} = \left( \frac{5 m_{1,2} V_A^2}{4k} \right)^{2/5}. \tag{27}
\]

Using the variables from Zener, Eq. (27) can be transformed into

\[
s_{max} = \left( \frac{5}{4} \right)^{2/5} TV_A. \tag{28}
\]

Hence, the simplified Hertzian force–displacement relation (25) results to

\[
F(s) = k \left( \frac{4}{5} \right)^{2/5} s_{max} s(t) \approx k \sqrt{0.915 \cdot s_{max} s(t)}. \tag{29}
\]
This simplified equation can be interpreted that the contact radius varies in dependence on the displacement \( s \). However, the maximum pressure is approximated as static and can be expressed with a constant displacement that is slightly smaller than the maximum displacement.

Likewise, the linearized Hertzian (Eq. (25)) can be derived from two linear springs in series connection where the springs represent both impacting bodies (sphere-sphere, sphere-plate). This can be used as an easily comprehensible alternative in spring-contact-models and thus, can simplify mathematical description within simulations.

With the simplified Hertzian equation, the differential equation describing the impact is as follows:

\[
S + \frac{1}{m_1} F'(s) + \sigma \frac{dF'(s)}{dt} = 0. \tag{30}
\]

In contrast to the analogous Eq. (10), Eq. (30) is a homogeneous damped oscillation equation which is analytical solvable. As done earlier for Eq. (10), Eq. (30) is similarly transformed into the dimensionless form

\[
d^2\sigma \frac{d\tau}{dt^2} + \left(1 + \frac{d}{dr}\right) \sigma = 0. \tag{31}
\]

The characteristic equation is \( x = d/d\tau \)

\[
x^2 + \lambda x + 1 = 0 \tag{32}
\]

and has the following solution

\[
x_{1,2} = \frac{-\lambda \pm \sqrt{\lambda^2 - 4}}{2}. \tag{33}
\]

Three cases of the solution are possible depending on the value of the discriminant \( D = \sqrt{\lambda^2 - 4} \), which can be imaginary, zero or larger than zero. For further evaluation, the first case is of main interest where the condition \( \lambda < 2 \) has to be satisfied. The general solution is given as

\[
\sigma = \exp\left(-\lambda \tau\right) \left(c_1 \sin \left(\hat{\omega} \tau\right) + c_2 \cos \left(\hat{\omega} \tau\right)\right), \tag{34}
\]

with

\[
\hat{\omega} = \frac{\lambda}{2} \tag{35}
\]

and

\[
\hat{\omega} = \frac{\sqrt{4 - \lambda^2}}{2}. \tag{36}
\]

With the first derivation

\[
\frac{d\sigma}{d\tau} = \exp\left(-\lambda \tau\right) \left(c_1 \hat{\omega} \cos \left(\hat{\omega} \tau\right) - \lambda \sin \left(\hat{\omega} \tau\right)\right) + c_2 \hat{\omega} \sin \left(\hat{\omega} \tau\right) - \lambda \cos \left(\hat{\omega} \tau\right)
\]

and the initial conditions (see Eq. (13)), the constants \( c_1 \) and \( c_2 \) can be exactly determined to

\[
c_1 = \hat{\omega}^{-1} \tag{38}
\]

and

\[
c_2 = 0. \tag{39}
\]

Thus, the unique solution

\[
\sigma = \hat{\omega}^{-1} \sin \left(\hat{\omega} \tau\right) \exp\left(-\lambda \tau\right) \tag{40}
\]

is obtained. With zero at the end of the rebound phase and the contact time (half period of sin-function) respectively

\[
\tau_{0,k} = \frac{\pi}{\hat{\omega}} \tag{41}
\]

The first derivation of Eq. (40)

\[
\frac{d\sigma}{d\tau} = \exp\left(-\lambda \tau\right) \left(\cos \left(\hat{\omega} \tau\right) - \hat{\omega}^{-1} \sin \left(\hat{\omega} \tau\right)\right) \tag{42}
\]

that corresponds to the velocity of the sphere and the initial condition from Eq. (23), the CoR can be calculated according to Eq. (24). With \( \hat{\omega} \tau = \pi \), the CoR thus reduces to

\[
e = \exp\left(-\pi \frac{\lambda}{\hat{\omega}}\right) \tag{43}
\]

and is rewritten with Eqs. (35) and (36) as

\[
e = \exp\left(-\pi \frac{\lambda}{\sqrt{4 - \lambda^2}}\right). \tag{44}
\]

Since \( \sigma = 1 \) has been selected, no real parameterization has been carried out. The same type of equation of CoR (Eq. (44)) is also used for linear viscoelastic models (Stronge, 2000; Schäfer et al., 1996; Schwager and Pöschel, 2007) which describe impact processes likewise simplified.

According to the model of Zener (Zener, 1941), the dependence of the CoR upon the inelasticity parameter is given in Fig. 1. It is shown that the CoR decreases with increasing values of the inelasticity parameter or increasing compliance of both impact partners, respectively. The CoR evaluated using the proposed analytical approach Eq. (44) also follows a decreasing trend with increasing values of the inelasticity parameter in analogy with the Zener model. A comparison between the evaluated CoR values with the original Zener model (Zener, 1941) and the proposed analytical approach Eq. (44) is shown in Figs. 1 and 2. The proposed analytical approach gives an acceptable approximation of the Zener model for the regime of inelasticity parameters \( \lambda < 1 \) and CoRs \( e \geq 0.2 \); only a slight deviation of the CoR exists \( (\Delta e \leq 0.015) \). However, the deviation of the total interval of CoRs \( (0 < e < 1) \) is within the range \( -0.038 \leq \Delta e \leq 0.015 \). Thus, for CoRs \( e < 0.2 \), the proposed analytical approach provides imprecise values. Nevertheless, since most unit operations (other than storage) of granular materials and particle products do not involve such collisions, it is logical to assume that the CoR remains higher than 0.2 and the inelasticity parameter remains within 1. Thus, the proposed analytical approach provides the needful applicability and feasibility, when the energy dissipation and the rebound velocity should be estimated for a (spherical) particle impacting on a large, thin plate, where the contact time is less than twice the transition time of elastic waves propagating across the thickness of the plate.

![Fig. 1. CoR in dependence upon the inelasticity parameter according to the Zener model (Zener, 1941) and comparison with the proposed analytical approach Eq. (44).](image-url)
4. Material samples

Industrially produced samples of elastic glass beads as well as elastic-plastic alumina ($\gamma$-$\text{Al}_2\text{O}_3$) and zeolite 4A granules of narrow size-distributions and high sphericity have been chosen.

The amorphous glass beads (produced by Sigmund Lindner GmbH) consist of soda-lime glass with 69.1% silica, 14.6% sodium oxide, 8.2% calcium oxide and smaller amounts of aluminium, magnesium, zinc and boric oxide (http://www.sigmund-lindner.com/produkte/silbeads-glaskugeln/typ-m.html). As the beads exhibit a smooth surface, a high degree of homogeneity, elastic material properties and are isotropic as well as (almost) monodisperse, they are suitable to be used as an ideal reference material for free-fall experiments.

The binderless $\gamma$-$\text{Al}_2\text{O}_3$ granules (produced by Sasol Germany GmbH) consist of approximately 98% of $\gamma$-$\text{Al}_2\text{O}_3$ and a small content of water as well as traces of carbon, iron-III-oxide, titanium oxide, sodium oxide and silicon oxide (http://www.sasoltechdata.com/alumina_group.asp).

The synthetic zeolite 4A granules (produced by Chemiewerk Bad Köstritz GmbH) consist of approximately 83 ma% zeolite primary particles ($\text{Na}_2\text{O}$$\cdot$$\text{Al}_2\text{O}_3$$\cdot$$2\text{SiO}_2$$\cdot$$n\text{H}_2\text{O}$) and 17 ma% attapulgite binder content ($\text{Mg,Al}_2\text{Si}_4\text{O}_{10}$($\text{OH}$)$_4$$\cdot$$2\text{H}_2\text{O}$) (Mueller et al., 2015).

The measured (most relevant) physical and granulometric characteristics of the material samples in the dry state are summarized in Table 1.

5. Free fall apparatus

The free fall tests had been performed using a home-built test rig, see Fig. 3. Fig. 4 presents the schematic representation of the particle movement during the conducted tests. At the beginning of each free fall test, one single particle is fixed using vacuum tweezers before being dropped so that it starts to fall freely without any initial velocity $v_0=0$ and rotation $\omega_0=0$. Different drop heights have been used to ensure different impact velocities in the range $0.3 \text{ m/s} < v_A < 2.5 \text{ m/s}$.

The vacuum-held particle (granule or glass bead) was released from the desired drop height $h_1$, such that it falls freely until impacting on the impact plate (glass plates of different thicknesses $H$, see Table 2) arranged orthogonal to the trajectory of the falling particle. Followed by the first impact, the particle rebounds until reaching a maximum rebound height $h_R$. Further on, the particle begins to fall again and a subsequent impact occurs on the plate. During each experiment, several impacts of the particle on the plate can occur before the particle has left the plate or remains motionless lying on the surface of the plate. The elastic waves generated at each impact, were recorded by a piezoelectric acceleration sensor, fixed at the reverse side of the impact plate. The impact and the rebound velocities had been determined from the time interval between consecutive impacts using Eq. (5) and thus, the CoR had been calculated using Eq. (6).

To obtain accurate and reproducible results, such that measurements of the impact and rebound events are precise, in

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**Table 1**

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Glass beads</th>
<th>$\gamma$-$\text{Al}_2\text{O}_3$</th>
<th>zeolite 4A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean diameter $d_{50}$ in mm</td>
<td>2.0 to 9.0</td>
<td>1.80</td>
<td>2.05</td>
</tr>
<tr>
<td>Sphericity $\psi$</td>
<td>$\geq 0.98$</td>
<td>0.99</td>
<td>0.98</td>
</tr>
<tr>
<td>Solid density $\rho_s$ in kg/m³</td>
<td>2500</td>
<td>3372</td>
<td>2179.9</td>
</tr>
<tr>
<td>Particle density $\rho_p$ in kg/m³</td>
<td>2500</td>
<td>879</td>
<td>1087.8</td>
</tr>
<tr>
<td>Porosity $\varepsilon$ in %</td>
<td>0</td>
<td>73.9</td>
<td>52.2</td>
</tr>
<tr>
<td>Modulus of elasticity $E$ in GPa</td>
<td>65.00</td>
<td>12.23</td>
<td>2.45</td>
</tr>
<tr>
<td>Yield pressure $p_y$ in MPa</td>
<td>-</td>
<td>756</td>
<td>288</td>
</tr>
<tr>
<td>Yield velocity $v_y$ in m/s</td>
<td>-</td>
<td>4.58</td>
<td>9.20</td>
</tr>
</tbody>
</table>

**Table 2**
Characteristics of the impact plate.

<table>
<thead>
<tr>
<th>Material name</th>
<th>Float glass</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modulus of elasticity $E$ in GPa</td>
<td>73</td>
</tr>
<tr>
<td>Density $\rho_s$ in kg/m³</td>
<td>2500</td>
</tr>
<tr>
<td>Dimensions $L/B/H$ in mm</td>
<td>300/300/1.9, 3.9, 7.9</td>
</tr>
</tbody>
</table>
addition to the meticulous attention given to the mechanical design, construction, alignment and calibration of test rig, much more than 100 tests have been carried out per series (considering particle type, particle size, drop height and impact plate thickness).

6. Results

The influence of different glass plate thicknesses \( (H=1.9 \text{ and } 7.9 \text{ mm}) \) on the CoR of the \( \gamma\text{-Al}_2\text{O}_3 \) \( (d_{50}=1.8 \text{ mm}) \) and zeolite 4A granules \( (d_{50}=2.05 \text{ mm}) \) is shown in Figs. 5 and 6. There is no appreciable influence of the plate thickness or the impact velocity \( (\text{in the measurement range } 0.3 \text{ m/s} < v_F < 2.5 \text{ m/s}) \) which can occur for example, during transportation, feeding, discharge as well as processing during processing in fluidized and spouted beds) on the CoR. The dominantly elastic-plastic \( \gamma\text{-Al}_2\text{O}_3 \) granules exhibit a CoR of \( e \approx 0.8 \) and the zeolite 4A granules a CoR of \( e \approx 0.7 \). Therefore, the energy dissipation is most likely due to the surface roughness of the granules, rotation at rebound and inelastic contact deformation (Mueller et al., 2015). However, the energy dissipation cannot be explained by elastic flexural waves.

This is due to the fact that bending in the form of flexural waves does not have a significant influence during the present impacts. This can be seen from the terms of the ratios of the densities as well as the ratios of the moduli of elasticity in the inelasticity parameter Eq. (20). The granules exhibit too small densities and too small moduli of elasticity compared to the glass plates, such that they could generate a significant amount of flexural waves at such ratios of particle diameter to plate thickness and such low impact velocities, respectively. In spite of the comparable large ratios of particle diameter to plate thickness \( (\text{considering values reported by Sondergaard (Sondergaard et al., 1990))}, \text{ in comparison with the theoretical Zener model (Zener, 1941), one can see that the inelasticity parameter of the granules at impact on the glass plates is very low (} \lambda < 0.028, \text{ see Table 3 and Fig. 1}). \text{ According to Zener (Zener, 1941), the theoretical CoR is near } e=1. \text{ Therefore, during impact, just a (negligible) small content of the kinetic energy of impact is transformed into elastic waves and dissipated, such that even multiple wave reflections hardly influence the impact.}

For this purpose, Figs. 7 and 8 summarize the theoretical curves of the CoRs of \( \gamma\text{-Al}_2\text{O}_3 \) and zeolite 4A granules for different impact velocities in dependence on the ratio of particle diameter to plate thickness according to the Zener model (Zener, 1941). However, for an accurate estimation of the CoR, a very small difference of \( \Delta e \approx -0.2 \text{ to } -0.3 \) also has to be taken into account due to several additional energy dissipations, see (Mueller et al., 2015). Moreover, the probability of energy dissipation by plastic yielding has to be considered for \( v_A > v_F \) (see yield velocity \( v_F \), given in Table 1). Furthermore, although the granules are highly spherical (see sphericity values in Table 1), unlike smooth glass beads, they may contain micro-defects or even flattened micro-asperities which upon contact with the impact plate may assume microscopically larger contact areas resulting in a slightly higher energy dissipation than predicted. Besides, the particle porosity could increase the energy dissipation due to a slightly higher damping and thereby probably, a delayed elastic restitution (of the porous structure). Due to such reasons, the predicted CoR may assume a very slightly over-estimated value than in reality.

![Fig. 5](image5.png) **Fig. 5.** Influence of the glass plate thickness \( (H=1.9 \text{ and } 7.9 \text{ mm}) \) on the CoR of dry \( \gamma\text{-Al}_2\text{O}_3 \) granules \( (d_{50}=1.8 \text{ mm}) \).

![Fig. 6](image6.png) **Fig. 6.** Influence of the glass plate thickness \( (H=1.9 \text{ and } 7.9 \text{ mm}) \) on the CoR of dry zeolite 4A granules \( (d_{50}=2.05 \text{ mm}) \).

![Table 3](table3.png) **Table 3** Selected ratios of particle diameter to plate thickness \( \gamma \), Eq. (22), and inelasticity parameter \( \lambda \), Eq. (20), of the material samples at impact on glass plates \( (H=1.9, 3.9 \text{ and } 7.9 \text{ mm}) \).
In comparison to Figs. 5 and 6 describing the CoRs of granules \((d_{50} = 2.05 \text{ mm})\) for different impact velocities \((v_A = 0.1, 1 \text{ and } 10 \text{ m/s})\) at impact on glass plates according to the Zener model \((\text{Zener, } 1941)\) and the proposed analytical approach Eq. (44), Figs. 9 and 10 show the CoRs of glass beads \((d_{50} = 4 \text{ to } 9 \text{ mm})\) versus glass plates \((H = 1.9 \text{ and } 3.9 \text{ mm})\).

Moreover, Fig. 11 shows the CoRs of the glass beads at impact on glass plates in dependence on the ratio of particle diameter to plate thickness \(\gamma = d / H\) at impact at a velocity of \(v_A = 1.34 \pm 0.015 \text{ m/s}\) and the theoretical curve according to the Zener model \((\text{Zener, } 1941)\) and the proposed analytical approach Eq. (44).

Fig. 8. Theoretical influence of the ratio of particle diameter to plate thickness \(\gamma\) on the CoR of dry zeolite 4A granules \((d_{50} = 2.05 \text{ mm})\) for different impact velocities \((v_A = 0.1, 1 \text{ and } 10 \text{ m/s})\) at impact on glass plates according to the Zener model \((\text{Zener, } 1941)\) and the proposed analytical approach Eq. (44).

Fig. 9. Influence of the particle (sphere) diameter on the CoRs of glass beads \((d_{50} = 4 \text{ to } 9 \text{ mm})\) at impact on a glass plate \((H = 1.9 \text{ mm})\) and the theoretical curves according to the Zener model \((\text{Zener, } 1941)\) and the proposed analytical approach Eq. (44).

Fig. 10. Influence of the particle (sphere) diameter on the CoRs of glass beads \((d_{50} = 4 \text{ to } 9 \text{ mm})\) at impact on a glass plate \((H = 3.9 \text{ mm})\) and the theoretical curves according to the Zener model \((\text{Zener, } 1941)\) and the proposed analytical approach Eq. (44).

Fig. 11. CoRs of glass beads \((d_{50} = 4 \text{ to } 9 \text{ mm})\) at impact on glass plates \((H = 1.9 \text{ and } 3.9 \text{ mm})\) and the theoretical curve according to the Zener model \((\text{Zener, } 1941)\) and the proposed analytical approach Eq. (44).

Fig. 12. CoRs of glass beads \((d_{50} = 4 \text{ to } 9 \text{ mm})\) at impact on glass plates \((H = 1.9 \text{ and } 3.9 \text{ mm})\), experimental values from Raman \((\text{Raman, } 1920)\) \((v_A = 2.34 \text{ m/s})\), the theoretical curves according to the Zener model \((\text{Zener, } 1941)\) and the proposed analytical approach Eq. (44).

Fig. 13. Theoretical influence of the ratio of particle diameter to plate thickness \(\gamma\) on the CoR of glass beads for different impact velocities at impact on glass plates according to the Zener model \((\text{Zener, } 1941)\) and the proposed analytical approach Eq. (44).
approach, the experimental values of Raman (Raman, 1920) have been added in Fig. 12. For an estimation of the CoR at impact of glass beads on glass plates, in Fig. 13, theoretical curves according to the Zener model have been drawn to illustrate the CoR versus the ratio of particle diameter to plate thickness for different impact velocities.

The theoretical curves and the experimental results exhibit the same trend of a decreasing CoR with increasing impact velocity. However, an almost constant deviation exists at which the experimental values remain below the theoretical curve. Also, no difference between the values of the two glass plates (H = 1.9 and 3.9 mm) as well as between different sphere diameters exist. An exact analysis of the measurements from Raman (Raman, 1920) reveals that measurements even with a given plate thickness exhibit a constant deviation to the Zener model. Therefore, no dependence on the sphere diameter exists. However, this deviation decreases with decreasing plate thickness until it is at a given plate thickness, even at almost zero.

The lower CoRs agree well with the results from Sondergaard et al. (1990), Tillett (1954) and Keller (1983), since the theoretical model only considers energy dissipation through elastic flexural waves (see for instance, Fig. 11). Additional processes are not considered, for example, dissipation due to the Hunter loss (energy dissipation into elastic spherically-symmetric outgoing wave within the impacting bodies) (Hunter, 1957), moisture on the surfaces, the surface roughness, friction, damping, plastic deformation, rotation of the sphere.

It should be noted that plastic yielding that limits the elastic material behavior does not occur in glass beads and glass plates. Also, additional energy dissipations due to crack initiation and growth do not develop below high impact velocities as confirmed by Kirchner and Gruver (1978) who observed the first Hertzian crack in glass beads (d = 3 mm) at an impact velocity of \( v_A \approx 30.6 \, \text{m/s} \).

According to Zener (1941), with increasing impact velocity and ratio of particle diameter to plate thickness, energy is increasingly dissipated into elastic flexural waves within the impact plate. Thereby, a decrease of the kinetic energy of the whole system does not necessarily take place, rather, an increasing content of kinetic energy of impact is converted into kinetic energy of the plate (Sondergaard et al., 1990). At high impact velocities and ratios of particle diameter to plate thickness, the vibrations of the glass plate are clearly noticeable and audible. At very high impact velocities and ratios of particle diameter to plate thickness, the total kinetic energy of impact may be converted into elastic waves where no rebound may occur, see Fig. 13.

On comparing the results of the granules and the glass beads, it becomes obvious that the ratio of particle diameter to plate thickness is not the only factor that has a considerable influence on the energy dissipation at impact on thin plates (with elastic waves reverting i.e. returning several times to the point of excitation located at the centre of the contact zone). Although the ratio is quadratically contained in the inelasticity parameter Eq. (20), the material combination of sphere and plate is also a significant factor. The kinetic energy of impact transmitted into the plates for formation of the Hertzian stress field and propagation of elastic waves depends upon the material properties. Whereas, the impact velocity with the dependency \( \lambda \sim \nu_0^{1/5} \) merely has a relatively lesser influence on the CoR.

7. Summary

An analytical approach to solve the otherwise mathematically laborious nonlinear differential equation of the Zener model has been presented, where the time-dependent displacement of the Hertzian force–displacement relation is just linearly contained. For the range of inelasticity parameters \( \lambda < 0.85 \) and CoRs \( \varepsilon > 0.2 \), the proposed analytical approach provides sufficiently accurate values. Thus, the approach is most suitable for application to study particle dynamics in process engineering endeavors involving impacts of coarse to ultrafine particle products.

Furthermore, the influence of the ratio of particle diameter to plate thickness and the impact velocity on the CoR of \( \gamma \)-Al2O3 and zeolite 4A granules has been studied. Since the inelasticity parameter remains very small during impact of granules on glass plates (see Table 3), the ratio of particle diameter to plate thickness (in spite of a sufficiently large ratio) has no significant influence on the CoR of the granules. The same is valid for the impact velocity if \( \nu_A < \nu_p \). Thus, merely a slight percentage of the kinetic energy of the impact is transformed into elastic waves.

However, for the elastic glass beads, a relation between CoR and the ratio of particle diameter to plate thickness as well as on the impact velocity and on the inelasticity parameter has been clearly observed and evaluated with the theoretical Zener model. One should note that the slight differences between the experimentally determined CoRs and the theoretically approached CoRs result due to several additional energy dissipations that are not considered by the model.

During impact of spheres on large thin plates, the ratio of sphere diameter to plate thickness and the material combination of sphere and plate are the factors that dominantly influence the CoR, whereas for \( \nu_A < \nu_p \), the impact velocity merely has a relatively lesser influence on the CoR.

Finite element method based comparative simulations of particle dynamics governed by the Zener model as well as the proposed analytical approach may be performed for a similar impact configuration of sphere and plate considering varying impact velocities and ratios of particle diameter to plate thickness. Similar study considering the impact configuration of cylinder and plate or cone and plate at which a piecewise linear force–displacement behavior occurs would also be of key relevance.

### Symbols

- \( B \): plate width (m)
- \( d \): particle diameter (mm)
- \( c' \): propagation velocity of quasi-longitudinal waves in thin plates (m/s)
- \( c_1, c_2 \): constants (dimensionless)
- \( E \): modulus of elasticity (GPa)
- \( E_{\text{kin}} \): kinetic energy (J)
- \( e \): coefficient of restitution (CoR) (dimensionless)
- \( F \): force (N)
- \( g \): gravitational acceleration (m/s²)
- \( H \): plate thickness (m)
- \( h \): drop height (m)
- \( k \): Hertzian stiffness constant (MPa \( \cdot \)m \( \frac{1}{2} \))
- \( L \): length (m)
- \( m \): mass (g)
- \( p \): pressure (GPa)
- \( R \): radius of surface curvature prior contact flattening (m)
- \( S \): displacement (mm)
- \( T \): constant of time (s)
- \( t \): time (s)
- \( v \): velocity (m/s)
- \( \alpha \): constant (dimensionless)
- \( \gamma \): ratio of particle diameter to plate thickness (dimensionless)
- \( \varepsilon \): porosity (dimensionless)
\[ \lambda \] inelasticity parameter (dimensionless)
\[ \omega \] angular velocity (rad/s)
\[ \lambda, \omega \] constants (dimensionless)
\[ \nu \] Poisson's ratio (dimensionless)
\[ \rho \] density (kg/m\(^3\))
\[ \sigma \] dimensionless displacement (dimensionless)
\[ \tau \] dimensionless time (dimensionless)
\[ \psi \] sphericity (dimensionless)
\[ \omega \] angular velocity (rad/s)

Indices

\[ A \] impact
\[ el \] elastic
\[ F \] plastic yielding
\[ g \] granule
\[ n \] nth impact
\[ p \] plate
\[ R \] rebound
\[ s \] solid
\[ 0 \] initial condition
\[ 1, 2 \] contact partner 1 (particle) and 2 (plate), 1st and 2nd impact
\[ 50 \] 50% quantil of cumulative size distribution

Acknowledgments

The financial support of this study by the German Research Foundation (DFG) within the framework of the project TO 156/40-1 "Statische und dynamische Beanspruchung elastischer, plastischer und viskoser Granulate" is gratefully acknowledged.

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